

Integrable evolution Hamiltonian equations of the third order with the Hamiltonian operator D_x

A.G. Meshkov¹ and V.V. Sokolov²

¹ Orel University – UNPK, Russia

² Landau Institute for Theoretical Physics, Moscow, Russia

ABSTRACT. All non-equivalent integrable evolution equations of third order of the form $u_t = D_x \frac{\delta H}{\delta u}$ are found.

1. INTRODUCTION.

We consider the third order integrable Hamiltonian evolution equations of the form

$$u_t = D_x \left(\frac{\delta H}{\delta u} \right) = D_x \left(\frac{\partial H}{\partial u} - D_x \frac{\partial H}{\partial u_x} \right). \quad (1.1)$$

Here $H(x, u, u_x)$ is the Hamiltonian and D_x is the total x -derivative. The celebrated KdV equation with $H = -\frac{1}{2}u_x^2 + \frac{1}{3}u^3$ provides the simplest example of such an equation. The function H is defined up to equivalence $H \rightarrow H + D_x f(x, u) + \lambda u$, where the function f and the constant λ are arbitrary.

Using the symmetry approach to integrability [1, 2], we obtain a complete list of canonical forms for integrable Hamiltonians H . Our proof of the classification statement contains an algorithm which allows to bring any integrable Hamiltonian to one of the canonical forms by canonical transformations.

1.1. Canonical transformations. Consider point transformations of the form

$$x = \varphi(y, v), \quad u = \psi(y, v). \quad (1.2)$$

The invertibility of the transformation is equivalent to the inequality $\Delta = \psi_v \varphi_y - \varphi_v \psi_y \neq 0$. Transformation (1.2) is called *canonical* if $\Delta = \psi_v \varphi_y - \varphi_v \psi_y = 1$. It is easy to verify that canonical transformations preserve the form of equation (1.1). The Hamiltonian of the resulting equation is given by

$$\tilde{H}(y, v, v_y) = H \left(\varphi(y, v), \psi(y, v), \frac{D_y(\psi)}{D_y(\varphi)} \right) D_y(\varphi). \quad (1.3)$$

Example. Linear transformations of the form

$$x = f(y), \quad u = \frac{v}{f'} + g(y), \quad \tilde{H} = H f'.$$

are canonical for arbitrary functions f and g .

Remark 1. If we consider only Hamiltonians that do not depend on x explicitly, we still have non-trivial canonical transformations

$$\begin{aligned} x &= f(v) + y, \quad u = v, & \tilde{H} &= (f' v_y + 1) H. \\ x &= v, \quad u = f(v) + y, & \tilde{H} &= H v_y. \quad \square \end{aligned}$$

Besides (1.2) we use the following canonical transformations of a more general form:

1. Dilatations of the form

$$t = \alpha \tilde{t}, \quad x = \beta y, \quad u = \gamma v, \quad \tilde{H} = \frac{\alpha}{\beta \gamma^2} H(\beta y, \gamma v)$$

are admissible for any H .

2. If H does not depend on x , then the Galilean transformation

$$y = x + ct, \quad v = u, \quad \tilde{H} = H - \frac{1}{2} c v^2;$$

is admissible.

3. If $H = c x u + h(u_x)$, where c is a constant, then the following transformation

$$u \rightarrow u + ct, \quad H \rightarrow H - c x u$$

is admissible.

1.2. Integrability conditions. Necessary integrability conditions for equations of the form

$$u_t = F(x, u, u_x, u_{xx}, u_{xxx}) \quad (1.4)$$

are given by a series of conservation laws [1, 2]:

$$\frac{d}{dt} \rho_n = \frac{d}{dx} \theta_n, \quad n = -1, 0, 1, \dots \quad (1.5)$$

where ρ_n are said to be the canonical densities. They can be defined by the following recursive formula presented here at the first time:

$$\begin{aligned} \rho_{n+2} = & \frac{1}{3} \rho_{-1} \left(\theta_n - F_0 \delta_{n,0} - F_1 \rho_n - F_2 D_x(\rho_n) - F_2 \sum_{-1}^n \rho_i \rho_j \right) - \frac{1}{3} (\rho_{-1})^{-2} D_x^2(\rho_n) \\ & - (\rho_{-1})^{-2} \left(\frac{1}{2} D_x \sum_{-1}^n \rho_i \rho_j + \frac{1}{3} \sum_0^n \rho_i \rho_j \rho_k + \rho_{-1} \sum_0^{n+1} \rho_i \rho_j \right), \quad n = -2, -1, 0, \dots, \end{aligned} \quad (1.6)$$

where $\rho_n = \theta_n = 0$ for $n < -1$, $F_n = \partial F / \partial u_n$, $\rho_{-1} = F_3^{-1/3}$, and δ_{ik} is the Kronecker symbol. By definition,

$$\sum_a^b \rho_{I_1} \cdots \rho_{I_k} = \sum_{\substack{I_s \geq a, 1 \leq s \leq k, \\ I_1 + \cdots + I_k = b}} \rho_{I_1} \cdots \rho_{I_k}.$$

In particular,

$$\begin{aligned} \sum_{-1}^{-2} \rho_i \rho_j &= \rho_{-1}^2, & \sum_{-1}^{-1} \rho_i \rho_j &= 2 \rho_{-1} \rho_0, & \sum_0^{-1} \rho_i \rho_j \rho_k &= 0, \\ \sum_0^0 \rho_i \rho_j \rho_k &= \rho_0^3, & \sum_0^1 \rho_i \rho_j \rho_k &= 3 \rho_0^2 \rho_1, \dots \end{aligned}$$

The first three canonical densities are given by

$$\begin{aligned}\rho_{-1} &= F_3^{-1/3}, \quad \rho_0 = -D_x(\ln \rho_{-1}) - \frac{1}{3}F_2\rho_{-1}, \\ \rho_1 &= \frac{1}{3}\theta_{-1}\rho_{-1} - \frac{1}{3}F_1\rho_{-1}^2 + F_2\rho_{-1}D_x(\rho_{-1}) + \frac{1}{9}F_2^2\rho_{-1}^5 + \frac{1}{3}\rho_{-1}^2D_x(F_2) \\ &\quad + \frac{2}{3}D_x(\rho_{-1}^{-2}D_x(\rho_{-1})) + \frac{1}{3}\rho_{-1}^{-3}(D_x(\rho_{-1}))^2.\end{aligned}$$

Notice that the flux θ_{-1} of the first canonical conservation law (1.5) is involved in the formula for ρ_1 .

The integrability conditions lead to some partial differential equations for the right hand side F of (1.4). We don't explain here how to derive these PDEs (see for example [3], where this technique is described in details).

For equations (1.1) any canonical density ρ_n can be expressed in terms of the Hamiltonian H and $\theta_{-1}, \theta_0, \dots, \theta_{n-3}, \theta_{n-2}$. In particular,

$$\rho_{-1} = - \left(\frac{\partial^2 H}{\partial u_1^2} \right)^{-1/3}.$$

Let us denote $\rho_{-1} = a$. Then

$$\frac{\partial^2 H}{\partial u_1^2} = -a^{-3}, \quad a = a(x, u, u_x).$$

The integrability conditions provide PDEs for H , which allow us to obtain a complete list of integrable Hamiltonians. It is known [1] that for Hamiltonian equations all even integrability conditions are trivial. Almost all the information of integrable Hamiltonians will be derived for the first and third integrability conditions.

2. CLASSIFICATION STATEMENT.

Theorem 1. *Any non-linear equation of the form (1.1) that has infinite hierarchy of higher symmetries*

$$u_{\tau_k} = F_k(x, u, u_x, \dots), \quad k = 1, 2, \dots$$

is canonically equivalent to one of the following equation:

$$u_t = D_x \left(\frac{u_{xx}}{a^3} - \frac{3a'}{2a^4}u_x^2 + \frac{\partial}{\partial u} \frac{P(u)}{a} \right), \quad H = -\frac{u_x^2}{2a^3} + \frac{P(u)}{a}, \quad (2.1)$$

$$\text{where } a = c_1 u^2 + c_2 u + c_3,$$

$$u_t = D_x \left(\frac{u_{xx}}{u^3} - \frac{3u_x^2}{2u^4} + P(x)u^2 \right), \quad H = -\frac{u_x^2}{2u^3} + \frac{1}{3}P(x)u^3, \quad (2.2)$$

$$u_t = D_x \left(\frac{u_{xx}}{\sqrt{u_x + P(u)}^3} + 3 \frac{P'(u)}{\sqrt{u_x + P(u)}} - \frac{P(u)P'(u)}{(u_x + P(u))^{3/2}} \right), \quad H = 4\sqrt{u_x + P(u)}. \quad (2.3)$$

Here P is an arbitrary polynomial of degree not greater than 4, c_i are arbitrary constants. \square

Remark 2. Using translations $u \rightarrow u + c$, dilatations $u \rightarrow \lambda u$, $t \rightarrow \alpha t$, $x \rightarrow \beta x$, and the Galilean transformation, one can reduce (2.1) to one of the following canonical forms:

$$u_t = D_x(u_{xx} + u^3), \quad H = -\frac{1}{2}u_x^2 + \frac{1}{4}u^4, \quad (2.1a)$$

$$u_t = D_x(u_{xx} + u^2), \quad H = -\frac{1}{2}u_x^2 + \frac{1}{3}u^3, \quad (2.1b)$$

$$u_t = D_x\left(\frac{u_{xx}}{u^3} - \frac{3u_x^2}{2u^4} + c_1u^2 + \frac{c_2}{u^2}\right), \quad H = -\frac{u_x^2}{2u^3} + \frac{1}{3}c_1u^3 - c_2u^{-1}, \quad (2.1c)$$

$$u_t = D_x\left(\frac{u_{xx}}{a^3} - 3\frac{uu_x^2}{a^4} + c_1\frac{c - u^2}{a^2} - 2c_2\frac{u}{a^2}\right), \quad H = -\frac{u_x^2}{2a^3} + \frac{c_1u + c_2}{a}, \quad (2.1d)$$

where $a = u^2 + c$. \square

Proof of Theorem 1. It follows from the first integrability condition ($n = -1$ in (1.5)) that

$$\frac{d}{dx}\left(a^3\frac{\partial^2 a}{\partial u_x^2}\right) = 0. \quad (2.4)$$

The solution of (2.4) is given by

$$a = \sqrt{a_1u_x^2 + a_2u_x + a_3}, \quad (2.5)$$

where $a_i = a_i(x, u)$ and

$$a_2^2 - 4a_1a_3 = \text{const}. \quad (2.6)$$

Under canonical transformations the function a transforms as follows

$$\tilde{a} = \sqrt{\tilde{a}_1v_y^2 + \tilde{a}_2v_y + \tilde{a}_3}, \quad \text{where } \tilde{a}_1 = a_1\varphi_x^2 - a_2\varphi_x\varphi_u + a_3\varphi_u^2.$$

Hence we can reduce a_1 to zero by an appropriate canonical transformation. Taking into account (2.6), we see that it suffices to consider the following two cases: **(A)** $a = a(x, u)$ and **(B)** $a = \sqrt{u_x + q(x, u)}$ (if $\tilde{a}_1 = 0$ then \tilde{a}_2 is a constant, which can be brought to 1 by a dilatation).

2.1. Case A. In this case the Hamiltonian is given by

$$H = h(x, u) - \frac{u_x^2}{2a^3}, \quad a = a(x, u). \quad (2.7)$$

The first integrability condition implies the following differential equations

$$\frac{\partial^3 a}{\partial u^3} = 0, \quad \frac{\partial}{\partial x}\left(\frac{\partial^2 a^2}{\partial u^2} - 3\left(\frac{\partial a}{\partial u}\right)^2\right) = 0,$$

therefore

$$a = s_1u^2 + s_2u + s_3, \quad s_i = s_i(x), \quad (2.8)$$

and

$$\frac{d}{dx}(s_2^2 - 4s_1s_3) = 0. \quad (2.9)$$

It is easy to verify that we can reduce the functions $s_1(x), s_2(x), s_3(x)$ to constants by a canonical transformation of the form

$$x = f(y), \quad u = \frac{v}{f'} + g(y).$$

So we obtain

$$H = h(x, u) - \frac{u_x^2}{2a^3}, \quad a = c_1 u^2 + c_2 u + c_3. \quad (2.10)$$

The third integrability condition implies

$$a \frac{\partial^5 h}{\partial u^5} + 5 a' \frac{\partial^4 h}{\partial u^4} + 10 a'' \frac{\partial^3 h}{\partial u^3} = 0.$$

Substituting g/a for h , we get $g^{(5)} = 0$ and therefore

$$H = \frac{r_1 u^4 + r_2 u^3 + r_3 u^2 + r_4 u + r_5}{c_1 u^2 + c_2 u + c_3} - \frac{u_x^2}{2(c_1 u^2 + c_2 u + c_3)^3}, \quad r_i = r_i(x). \quad (2.11)$$

To determine the x -dependence of the functions $r_i(x)$ we consider several subcases.

Subcase A.1. Let $c_1 = c_2 = 0$. Then the Hamiltonian is equivalent to

$$H = -\frac{1}{2} u_x^2 + \frac{1}{4} q_1 u^4 + \frac{1}{3} q_2 u^3 + \frac{1}{2} q_3 u^2 + q_4 u,$$

where $q_i = q_i(x)$. The third integrability condition is equivalent to relations

$$q_1 = k_1, \quad 2 q_2 q_2' = 3 k_1 q_3', \quad 2 q_2''' + 2 q_2' q_3 = 4 k_1 q_4', \quad (2.12)$$

where k_1 is a constant.

If $k_1 \neq 0$, then we normalize it to 1 by $u \rightarrow u k_1^{-1/2}$ and reduce q_2 to zero by $u \rightarrow u - q_2(x)/3$. It follows from (2.12) that now we have $q_3' = q_4' = 0$. Thus the Hamiltonian is equivalent to (2.1a) and we arrive at the modified KdV equation.

If $k_1 = 0$, then (2.12) implies $q_2 = k_2$, where k_2 is a constant. Suppose that $k_2 \neq 0$. Then we normilize k_2 to 1 and reduce q_3 to zero by $u \rightarrow u - q_3(x)/2$. It follows from the fifth integrability condition that $q_4' = 0$ and we obtain the KdV equation (2.1b).

At last, if $k_1 = k_2 = 0$, then corresponding equation becomes linear:

$$u_t = D_x(u_{xx} + f(x)u + g(x)). \quad (2.13)$$

where f and g are arbitrary functions.

Subcase A.2. Suppose $c_1 = 0, c_2 \neq 0$. Then the Hamiltonian is equivalent to

$$H = -\frac{u_x^2}{2u^3} + \frac{1}{3} q_1 u^3 + \frac{1}{2} q_2 u^2 - q_3 u^{-1} + q_4 u,$$

where q_1, q_2, q_3 and q_4 are functions of x . It follows from the third integrability condition that

$$q_1' q_2 - 2 q_1 q_2' = 0, \quad q_2''' - q_2 q_3' - 2 q_2' q_3 = 0, \quad q_4' = 0. \quad (2.14)$$

Since q_4 is a constant, without loss of generality we put $q_4 = 0$.

Under canonical transformation

$$y = f(x), \quad v = u/f'$$

the Hamiltonian transforms as follows:

$$\tilde{H} = -\frac{v_y^2}{2v^3} + \frac{1}{3}q_1 f'^2 v^3 + \frac{1}{2}q_2 f' v^2 - (v f'^4)^{-1} \left(f' f''' - \frac{3}{2} f''^2 + q_3 f'^2 \right).$$

Consider the following cases: **(a)** $q_2 \neq 0$ and **(b)** $q_2 = 0$.

In the case **(a)** taking $\int 1/q_2 dx$ for f , we bring q_2 to 1. Then it follows from (2.14) that $q'_1 = q'_3 = 0$ and we get an equation equivalent to (2.1c).

In the case **(b)** taking for f any nonconstant solution of equation $f' f''' - \frac{3}{2} f''^2 + q_3 f'^2$, we reduce q_3 to zero. Then the fifth integrability condition implies $q_1^{(5)} = 0$ and we arrive at equation (2.2).

Remark 3. If we consider the normalization $q_3 = 0$ instead of $q_2 = 1$ in the case **(a)**, then we get

$$u_t = D_x \left(\frac{u_{xx}}{u^3} - \frac{3u_x^2}{2u^4} + c q_2^2(x) u^2 + q_2(x) u \right), \quad H = -\frac{u_x^2}{2u^3} + \frac{1}{3} c q_2^2(x) u^3 + \frac{1}{2} q_2(x) u^2, \quad q_2''' = 0.$$

We have chosen the canonical form (2.1c) since the corresponding Hamiltonian does not depend on x explicitly. \square

Remark 4. If we consider the normalization $q_1 = 1$ instead of $q_3 = 0$ in the case **(b)**, then we obtain

$$u_t = D_x \left(\frac{u_{xx}}{u^3} - \frac{3u_x^2}{2u^4} + u^2 + \frac{3}{2} \frac{\wp(x)}{u^2} \right),$$

where $\wp'^2 = 4\wp^3 - g_2\wp - g_3$, $\wp' \neq 0$. It is another canonical form for equation (2.2). \square

Subcase A.3. Suppose $c_1 \neq 0$ in (2.11). Using the dilatation $u \rightarrow u c_1^{-1/2}$, we normalize c_1 to 1. Then the translation $u \rightarrow u - c_2/2$ reduces a to the form $a = u^2 + c$. In this case the first integrability condition yields $\partial H / \partial x = 0$. Therefore all the functions r_i in (2.11) are constants and the equation is equivalent to (2.1d).

2.2. Case B. Consider Hamiltonians of the form

$$H = h(x, u) + 4a, \quad a = \sqrt{u_x + q(x, u)}.$$

It follows from the first integrability condition that

$$\frac{\partial^3 h}{\partial u^3} = 0, \tag{2.15}$$

$$\frac{\partial^3 h}{\partial x^2 \partial u} - 2q \frac{\partial^3 h}{\partial x \partial u^2} + \frac{\partial q}{\partial u} \frac{\partial^2 h}{\partial x \partial u} - \frac{\partial q}{\partial x} \frac{\partial^2 h}{\partial u^2} = 0. \tag{2.16}$$

The third integrability condition implies one more simple PDE: $\partial^5 q / \partial u^5 = 0$. Solving this equation and (2.15) we find that

$$q = q_1 u^4 + q_2 u^3 + q_3 u^2 + q_4 u + q_5, \quad h = \frac{1}{2} h_1 u^2 + h_2 u + h_3, \tag{2.17}$$

where $q_i = q_i(x)$, $h_i = h_i(x)$. Substituting q and h in (2.16) we obtain the following system:

$$\begin{aligned} 2q_1 h'_1 - q'_1 h_1 &= 0, & q_2 h'_1 - q'_2 h_1 + 4q_1 h_2 &= 0, & q'_3 h_1 - 3q_2 h'_2 &= 0, \\ h''_1 + 2q_3 h'_2 - q_4 h'_1 - q'_4 h_1 &= 0, & h''_2 - 2q_5 h'_1 + q_4 h'_2 - q'_5 h_1 &= 0. \end{aligned} \tag{2.18}$$

The canonical transformation

$$y = \varphi(x), \quad v = \frac{u}{\varphi'} + \psi(x),$$

changes the Hamiltonian as follows:

$$\tilde{H} = \frac{1}{2} \tilde{h}_1 v^2 + \tilde{h}_2 v + \tilde{h}_3 + 4 \sqrt{v_y + \tilde{q}},$$

where

$$\tilde{h}_1 = \varphi' h_1, \quad \tilde{h}_2 = h_2 - \varphi' \psi h_1, \quad \tilde{q} = Q + (\varphi')^{-2} (\varphi'' v - (\psi \varphi')'), \quad (2.19)$$

$$Q = q_1 (\varphi')^2 (v - \psi)^4 + q_2 \varphi' (v - \psi)^3 + q_3 (v - \psi)^2 + q_4 (\varphi')^{-1} (v - \psi) + q_5 (\varphi')^{-2}.$$

If $h_1 \neq 0$ then we put $\varphi' = 1/h_1$ and $\psi = h_2$ to get $h_1 = 1$ and $h_2 = 0$. Now it follows from (2.18) that $\partial q_i / \partial x = 0$. Since H does not depend on x we remove the term $\frac{1}{2} u^2$ in H by the Galilean transformation and obtain equation (2.3).

If $h_1 = 0$ but $h'_2 \neq 0$ then equations (2.18) lead to $q_1 = q_2 = q_3 = 0$, $h''_2 + q_4 h'_2 = 0$. It follows from the formula

$$\tilde{q} = \frac{v}{\varphi'^2} (\varphi'' + q_4 \varphi') + \frac{1}{\varphi'^2} (q_5 - (\psi \varphi')')$$

that there exist φ and ψ such that $\tilde{q} = 0$. In this case $h_2 = cx$, where c is a constant. So, we obtain

$$H = c x u + 4 \sqrt{u_x}, \quad u_t = D_x \left(\frac{u_{xx}}{u_x^{3/2}} + c x \right).$$

Using the transformation $u \rightarrow u + ct$, we bring c to zero and arrive at a particular case of equation (2.3).

If $h_1 = h'_2 = 0$ then without loss of generality we put $h_2 = h_3 = 0$. Thus, we have shown that in all cases the functions h_1, h_2, h_3 can be reduced to zeros.

Now we normalize the polynomial q prove that all coefficients of q can be reduced to constants. If $q_1 \neq 0$ we use the normalization $q_1 = 1$, $q_2 = 0$. If $q_1 = 0$, $q_2 \neq 0$ then we normalize q_2 and q_3 by 1 and 0 correspondingly. In the case $q_1 = q_2 = 0$ we may put $q_4 = q_5 = 0$. In each of these cases the third integrability condition gives rise to $q'_i = 0$ for all remaining coefficients of q . The corresponding equations can be obtained from (2.3) by translations $u \rightarrow u + c$ and dilatations $u \rightarrow \lambda u$, $t \rightarrow \alpha t$, $x \rightarrow \beta x$. \square

2.3. Integrability of equations (2.1)–(2.3). Equations (2.1a) and (2.1b) are known to be integrable by the inverse scattering method. Equations (2.1c), (2.1d) and (2.3) can be reduced to known integrable equations of the form [4]

$$v_t = v_{yyy} + G(x, v, v_y, v_{yy}) \quad (2.20)$$

by the standard reciprocal transformation (see [2], section 1.4)

$$dy = \rho_{-1} dx + \theta_{-1} dt, \quad v(t, y) = u(t, x), \quad (2.21)$$

where ρ_{-1} is the first canonical density and θ_{-1} is the correspondent flux. Notice that this transformation is always applicable if ρ_{-1} depends on u only and the r.h.s. of the equation does

not depend on x . Sometimes (2.21) can be applied in the case when ρ_{-1} depends on u and u_x . That is the case for equation (2.3).

To reduce the equation for v to an usual form some additional point transformation $v = f(w)$ can be needed. For equation (2.1c) we have $\rho_{-1} = u$. Taking $v = e^w$, we obtain

$$w_t = w_{yyy} - \frac{1}{2}w_y^3 + w_y (c_1 e^{2w} - 3c_2 e^{-2w}).$$

This equation was found by F. Calogero and A. Degasperis and independently by A. Fokas.

In the case of equation (2.1d) we have $\rho_{-1} = u^2 + c$. If $c \neq 0$ we put $c = -k^2/4$, $v = \frac{k}{2} \tanh(w/2)$. As a result we get the same (up to the Galilean transformation) Calogero–Degasperis equation

$$w_t = w_{yyy} - \frac{1}{2}w_y^3 + w_y (\tilde{c}_1 e^{2w} + \tilde{c}_2 e^{-2w}) - c_3 w_y,$$

where $\tilde{c}_1 = 3/2 k^{-2}(2c_2 + k c_1)$, $\tilde{c}_2 = 3/2 k^{-2}(2c_2 - k c_1)$, $c_3 = 6c_2 k^{-2}$. In the case $c = 0$ we put $v = 1/w$ to obtain the mKdV equation:

$$w_t = w_{yyy} + 12c_2 w^2 w_y + 6c_1 w w_y.$$

Equation (2.3) is related to one more equation found by Calogero and Degasperis:

$$v_t = v_{yyy} - \frac{3}{8} \frac{(D_y(Q + v_y^2))^2}{v_y(v_y + Q)} + \frac{1}{2} Q'' v_y, \quad Q = 4P,$$

by the transformation (2.21).

Since the righthand side of equation (2.2) depends on x we can not apply transformation (2.21) straightforwardly. Instead we perform the substitution $u \rightarrow u_x$ to get the potential form

$$u_t = \frac{u_{xxx}}{u_x^3} - \frac{3u_{xx}^2}{2u_x^4} + P(x)u_x^2.$$

The hodograph transformation $y = u(t, x)$, $v(t, y) = x$ brings the latter equation to the Krichever – Novikov equation

$$v_t = v_{yyy} - \frac{3v_{yy}^2}{2v_y} - \frac{P(v)}{v_y}.$$

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